Rank of a Matrix :-

A number r is said to be Rank of a matrix A if the following properties are satisfied

i) There is atleast one square submatrix of A of order r whose determinant is not zero.

ii) If the matrix A contains any square submatrix of order $r+1$, then the determinant of every square submatrix of A of order $r+1$ should be zero.

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -3 \end{pmatrix}$$

Find the rank for the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \quad 3$$

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ -2 & 0 & 1 \end{vmatrix} = 2(1-0) - 3(0+4) + 4(0+2)$$

$$= 2(1) - 3(4) + 4(2)$$

$$= 2 - 12 + 8$$

$$= -2 \neq 0$$

$$\therefore \boxed{\rho(A) = 3}$$

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2) $B = \begin{pmatrix} 3 & 4 & -3 \\ 5 & 5 & -3 \\ 2 & 1 & 0 \end{pmatrix}$

$$|B| = 3(0+3) - 4(0+6) - 3(5-10)$$
$$= 9 - 24 + 15$$
$$= 0.$$

$\therefore \rho(B) \neq 3.$

Consider, $\begin{vmatrix} 3 & 4 \\ 5 & 5 \end{vmatrix} = 15 - 20 = -5 \neq 0.$ (Submatrix of B of order 2)

$\therefore \boxed{\rho(B) = 2}$

3) $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ 2 & -2 & 0 & 1 \\ 3 & 2 & 1 & 3 \end{pmatrix}$

$A \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 6 \\ 0 & -6 & -6 & -7 \\ 0 & -4 & -8 & -9 \end{pmatrix}$ $\begin{array}{l} R_2 \to R_1 + R_2 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1 \end{array}$

$A \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 6 \\ 0 & 6 & 6 & 7 \\ 0 & 4 & 8 & 9 \end{pmatrix}$ $\begin{array}{l} R_3 \to R_3/-1 \\ R_4 \to R_4/-1 \end{array}$

$\sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 6 \\ 0 & 6 & 6 & 7 \\ 0 & 0 & 0 & -3 \end{pmatrix}$ $\begin{array}{ccc} -6 & -12 & -18 \\ 6 & 6 & 7 \\ \hline 0 & -6 & -11 \end{array}$

$R_4 \to R_4 - 2R_3$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 6 \\ 0 & 6 & 6 & 7 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -6 & -11 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{matrix} R_2 \to R_2/2 \\ R_3 \to R_3 - 3R_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 6 & 11 \\ 0 & 0 & 0 & -3 \end{pmatrix} R_3 \to R_3/-1$$

$$\Rightarrow A \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 6 & 11 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

no. of non-empty rows $= 4$.

$\therefore \boxed{\rho(A) = 4}$

4. $A = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & 1 & -1 & 2 \\ 4 & 0 & 1 & 0 & 3 \\ 9 & -1 & 2 & 3 & 7 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 4 & 1 & -8 & -1 \\ 0 & 8 & 2 & -15 & 6 \end{pmatrix}$$

$$\begin{array}{cccc} -8 & -2 & 14 & 2 \\ 8 & 2 & -15 & \\ \hline 0 & 0 & -1 & 8 \end{array}$$

$$\sim \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 4 & 1 & -8 & -1 \\ 0 & 8 & 2 & -15 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{5\times 4}$$

Here, non-empty rows = 3

$\boxed{\rho(A) = 3}$

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Characteristic roots and characteristic vectors of a matrix :-

Definition :-

Let $A = (a_{ij})$ be an 'n' rowed square matrix and λ be a scalar. Then the matrix $A - \lambda I$ is called characteristic matrix of A also the determinant.

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

is an ordinary polynomial of λ of degree n and is called the characteristic polynomial of A.

The solution of the equation $|A - \lambda I| = 0$ are called characteristic values or characteristic eigen values of the matrix A.

Eigen vectors or characteristic vectors:-
The solution of $|A - \lambda I| = 0$ are called eigen vectors or characteristic vectors.

1. Determine the characteristic roots or eigen values of the matrix.

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

characteristic equation is $|A - \lambda I| = 0$

$$\left| \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\left|\begin{pmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{pmatrix}\right| = 0$$

$$\Rightarrow -\lambda(\lambda^2-1)-1(-\lambda+2)+2(-1+2\lambda)=0$$

$$-\lambda^3+\lambda+\lambda-2-2+4\lambda=0$$

$$-\lambda^3+6\lambda-4=0$$

$$\lambda^3-6\lambda+4=0$$

$$\begin{array}{c|cccc} 2 & 1 & 0 & -6 & 4 \\ & 0 & 2 & 4 & -4 \\ \hline & 1 & 2 & -2 & 0 \end{array}$$

$$(\lambda-2)(\lambda^2+2\lambda-2)=0$$

$$(\lambda-2)=0 \qquad \lambda^2+2\lambda-2=0$$

$$\boxed{\lambda=2} \qquad (\lambda+1)^2-1-2=0$$

$$(\lambda+1)^2=3$$

$$\lambda+1=\pm\sqrt{3}$$

$$\boxed{\lambda=-1\pm\sqrt{3}}$$

$\therefore$ The Eigen values of the matrix A are
$2, -1+\sqrt{3}, -1-\sqrt{3}$

2\. $A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{pmatrix}$

$$|A-\lambda I|=0$$

$$\begin{vmatrix} 1-\lambda & -1 & 2 \\ -2 & 1-\lambda & 3 \\ 3 & 2 & -3-\lambda \end{vmatrix}=0$$

$$(1-\lambda)[(1-\lambda)(-3-\lambda)-6]+1(6+2\lambda-9)+2(-4-3+3\lambda)=0$$

$$(1-\lambda)[-3-\lambda+3\lambda+\lambda^2-6]+(2\lambda-3)+2(3\lambda-7)=0$$

$$(1-\lambda)(-9+2\lambda+\lambda^2)+2\lambda-3+6\lambda-14=0$$

$-9+2\lambda+\lambda^2+9\lambda-2\lambda^2-\lambda^3+2\lambda-3+6\lambda-14=0$

$-\lambda^3-\lambda^2+19\lambda-26=0$

$\lambda^3+\lambda^2-19\lambda+26=0$

$$\begin{array}{r|rrrr} 2 & 1 & 1 & -19 & 26 \\ & 0 & 2 & 6 & -26 \\ \hline & 1 & 3 & -13 & 0 \end{array}$$

$(\lambda-2)=0$ | $\lambda^2+3\lambda-13=0$

$\lambda=2$ | $\lambda^2+3\lambda-13=0$

$\left(\lambda+\frac{3}{2}\right)^2-\frac{9}{4}-\frac{13}{2}=0$

$\left(\lambda+\frac{3}{2}\right)^2-\frac{9-52}{4}=0$

$\left(\lambda+\frac{3}{2}\right)^2=\frac{61}{4}$

$\lambda=-\frac{3}{2}\pm\frac{\sqrt{61}}{2}$

$\therefore$ The Eigen values of the matrix A are

$2,\ -\frac{3}{2}+\frac{\sqrt{61}}{2},\ -\frac{3}{2}-\frac{\sqrt{61}}{2}$

$\frac{13\times 8}{104}$ $\frac{13\times 4}{52}$

$(1-\lambda)(1-\lambda)(1-\lambda)$ $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ $=I$

$(1-\lambda)(1-2\lambda+\lambda^2)$

$1-2\lambda+\lambda^2-\lambda+2\lambda^2-\lambda^3$

$1-3\lambda+3\lambda^2-\lambda^3$

Shortcut method :-

$\lambda^3-\lambda^2$(Sum of the diagonal elements)

$+\lambda$(Sum of the co-factors of diagonal elements)

$-\det A=0$ to find the cubic equation

(for $|A-\lambda I|=0$).

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3. Determine Eigen Values and Eigen Vector of $A=\begin{pmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{pmatrix}$.

$|A|=8(5)+6(-10)+2(10)$

$=40-60+20$

$=0$

$\lambda^3-\lambda^2(8+7+3)+\lambda(5+20+20)-0=0$

$\lambda^3-18\lambda^2+45\lambda=0$

$\Rightarrow \lambda=0$ | $\lambda^2-18\lambda+45=0$

$(\lambda-9)^2-81+45=0$

$(\lambda-9)^2=36$

$\lambda-9=\pm 6$

$\lambda=9\pm 6$

$\Rightarrow \lambda=3,15.$

$(21-16)+(20)+(20)$

$5+20+20$

$=55$

$\frac{81}{45}{36}$

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$\therefore$ The Eigen values of the matrix A are
0, 3, 15

To find the eigen vectors:-
Now, we solve the eqn $(A-\lambda I)X=0$

ie $\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \longrightarrow (1)$

Case (i) :- put $\lambda = 0$ in (1)

$\Rightarrow \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$8x_1 - 6x_2 + 2x_3 = 0 \longrightarrow (2)$
$-6x_1 + 7x_2 - 4x_3 = 0 \longrightarrow (3)$
$2x_1 - 4x_2 + 3x_3 = 0 \longrightarrow (4)$

$(3) \Rightarrow \cancel{-6x_1} + 7x_2 - 4x_3 = 0$
$3\times(4) \Rightarrow \cancel{6x_1} - 12x_2 + 9x_3 = 0$

$-5x_2 + 5x_3 = 0$

$5x_3 = 5x_2$

$\boxed{x_3 = x_2}$

put $x_3 = x_2$ in (2)

$(2) \Rightarrow 8x_1 - 4x_2 = 0$

$8x_1 = 4x_2$

$\Rightarrow \boxed{2x_1 = x_2}$

put $x_3 = x_2 = 1 \Rightarrow x_1 = 1/2$

(ie) $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}$ (or) $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

case (ii) put $\lambda = 3$ in ①

$$④ \Rightarrow \begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$5x_1 - 6x_2 + 2x_3 = 0 \longrightarrow ⑤$$
$$-6x_1 + 4x_2 - 4x_3 = 0 \longrightarrow ⑥$$
$$2x_1 - 4x_2 = 0$$
$$\Rightarrow 2x_1 = 4x_2$$
$$\boxed{x_1 = 2x_2}$$

put in ⑥

$$⑥ \Rightarrow -6x_1 + 2(x_1) - 4x_3 = 0.$$
$$-4x_1 - 4x_3 = 0$$
$$\boxed{x_1 = -x_3}$$

put $x_2 = 1 \Rightarrow x_1 = 1/2,\ x_3 = -1/2$.

$$\Rightarrow X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

case (iii) put $\lambda = 15$ in ①

$$④ \Rightarrow \begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-7x_1 - 6x_2 + 2x_3 = 0 \rightarrow ⑧$$
$$-6x_1 - 8x_2 - 4x_3 = 0 \rightarrow ⑨$$
$$2x_1 - 4x_2 - 12x_3 = 0 \rightarrow ⑩.$$

from eqn ⑧ & ⑨

$$\frac{x_1}{24+16} = \frac{-x_2}{28+12} = \frac{x_3}{56-36}$$

1 2 −1

$\frac{1}{-1}$ 1/2

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$$\frac{x_1}{40} = \frac{-x_2}{40} = \frac{x_3}{20}$$

$\div 40$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1/2}$$

$\Rightarrow$ The eigen vector,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1/2 \end{pmatrix} //.$$

25/7/17.

4: $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

$2(3)+1(-2+1)$
$+1(1-2)$
$6-1-1$

$$\lambda^3 - \lambda^2(2+2+2) + \lambda(3+3+3) - 4 = 0$$

$$\lambda^3 - 6\lambda^3 + 9\lambda - 4 = 0$$

$$(\lambda-1)(\lambda^2 - 5\lambda + 4) = 0$$

1	1	-6	9	-4
	0	1	-5	4
	1	-5	4	0

$\lambda = 1$ | $\lambda^2 - 5\lambda + 4 = 0$

$(\lambda-4)(\lambda-1) = 0$

$\lambda = 4, \lambda = 1$

4, −4, −1, −5

The Eigen values of the matrix A is
$\lambda = 1, 4, 1.$

To fing the eigen vectors:-

$(A - \lambda I)x = 0$

ie $\begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \longrightarrow (1)$

Case(i) :- $\lambda = 1$

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$x_1 - x_2 + x_3 = 0 \longrightarrow (2)$
$-x_1 + x_2 - x_3 = 0 \longrightarrow (3)$
$x_1 - x_2 + x_3 = 0 \longrightarrow (4)$
(identical equations)

Let us assume,
$x_1 = 1, x_2 = 2, x_3 = 1$ which satisfies
the above eqns.

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

case 2: $\lambda = 4$

$$(1) \Rightarrow \begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$-2x_1 - x_2 + x_3 = 0 \longrightarrow (5)$
$-x_1 - 2x_2 - x_3 = 0 \longrightarrow (6)$
$x_1 - x_2 - 2x_3 = 0 \longrightarrow (7)$

$(5) + (6) \Rightarrow -3x_1 - 3x_2 = 0$
$-3x_1 = 3x_2$
$\boxed{x_1 = -x_2}$

put in (7)
$\Rightarrow x_1 + x_1 - 2x_3 = 0$
$2x_1 = 2x_3$
$\boxed{x_1 = x_3}.$

Let us assume, $x_1 = 1, x_2 = -1, x_3 = 1$
which satisfies the above eqns.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$